Research Statement - Lior Alon

My research lies in mathematical physics and intersects with various mathematical disciplines, including *Spectral Geometry, Graph Theory, Morse Theory, Real Algebraic Geometry, Fourier Analysis, Dynamics, and Number Theory.* My papers can be divided into three subjects:

1. **Fourier Quasicrystals and Lee-Yang (stable) polynomials** [7, 9]
2. **Spectral geometry of quantum graphs** [2–6]
3. **Nodal edge count and Morse theory of magnetic operators on discrete graphs** [8]

We begin with a bird's-eye view of each subject. More details are later.

0.1 Fourier Quasicrystals

Crystals are periodic structures. The Poisson summation formula captures periodicity via Fourier transform, stating that it transforms the counting measure of any lattice to the counting measure of its dual lattice. A *Fourier Quasicrystal (FQ)* resembles a periodic lattice in the sense that it is a measure supported on a discrete set whose Fourier transform is also a measure supported on a discrete set.

Kurasov and Sarnak recently provided a novel construction of non-periodic one-dimensional FQs by intersecting irrational lines with periodic hypersurfaces of certain kind, see Figure 1 Left. We proved that every FQ with integer weights can be constructed this way, and used it to describe the distribution of gaps between atoms. In an ongoing work, we extend this construction to FQs of any dimension d, by intersecting d-dimensional vector spaces with certain codimension-d algebraic varieties, see Figure 1.

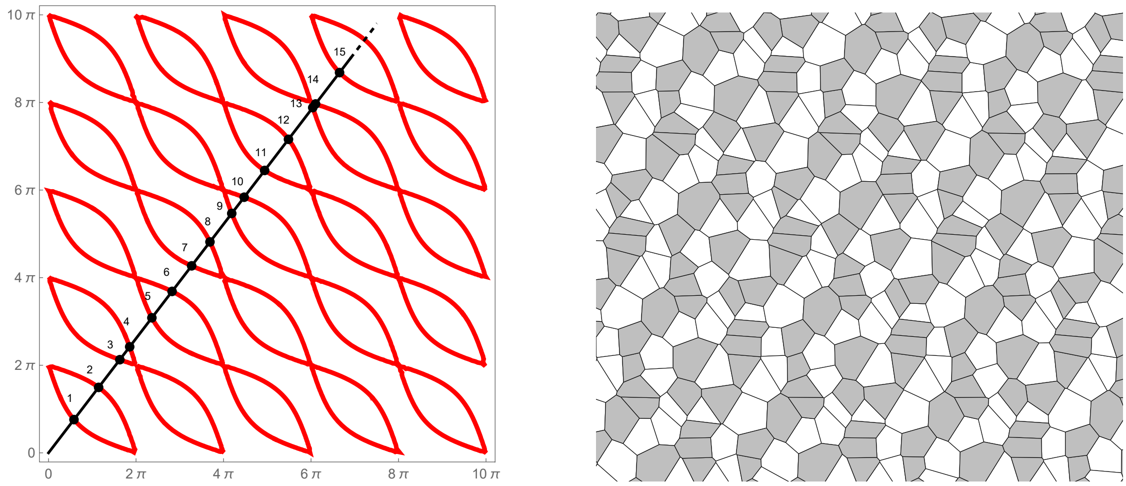

Figure 1: (Left) Construction of 1D Fourier Quasicrystal. (Right) A Voronoi diagram of a 2D Fourier Quasicrystal.

0.2 Spectral geometry, quantum chaos, and nodal count of quantum graphs

Spectral Geometry explores the relationship between the geometry of an object and the wave patterns and frequencies it supports, represented by the eigenfunctions (wave-functions) and eigenvalues (energy levels) of the associated Laplacian. While for simple objects like a line segment (quantum infinite well), the differential equations can be explicitly solved, this isn't generally the case. We often settle for qualitative descriptions of solutions and their average properties. However, there is balm as well as bitterness, the geometry of an object becomes more complex, its associated spectrum and wave-functions often exhibit universal properties, which makes them easier to study. The field of *quantum chaos* focuses on such phenomena.

A *quantum graph*, or metric graph, comprises a network of line segments of varying lengths, connected at their endpoints following a graph structure. The Laplacian acts as the second derivative on functions on the segments that satisfy certain boundary conditions at each vertex, resulting in a system of coupled ODEs. The quantum graph (by which we mean the Laplacian on it) possesses infinitely many eigenvalues and eigenfunctions. This model strikes a balance between simplicity and complexity, as eigenfunctions can be explicitly expressed, but with $\mathbb{Q}$-linearly independent edge lengths and sufficiently large graph structures, universal phenomena associated with quantum chaos emerge.

Our work shows that such universal phenomena manifest in the fluctuations of the *nodal count* sequence. Here, the nodal count sequence stands for the number of points where the k-th eigenfunction vanish, as $k \to \infty$. It is known to equal to $k-1$ plus a positive deviation referred to as *nodal surplus*, which is bounded by β, the first Betti number of the graph. We demonstrated numerically that for sufficiently large graphs, the nodal surplus tends to a Gaussian centered at $\beta/2$ and we proved convergence to a Gaussian for several graph families, as predicted.

These results and others are based on the analysis of a moduli space encompassing all quantum graphs associated with a given graph structure. This space was recently proven to be irreducible by Kurasov and Sarnak. In a separate study, I leveraged this irreducibility to analyze generic eigenfunctions and the spectrum's rigidity. Specifically, I demonstrated that if two quantum graphs share the same $\mathbb{Q}$-linearly independent edge lengths but different graph structures, they cannot share common eigenvalues, except for a possibly zero-density sub-sequence.

0.3 Nodal edge count and Morse theory of magnetic operators on discrete graphs

The discrete Laplacian H of a graph G with n edges is the $n \times n$ matrix with H_{ii} equal to the degree of the i-th vertex, $H_{ij} = -1$ when (i, j) is an edge, and $H_{ij} = 0$ otherwise. The *nodal edge count* of an eigenvector v of H is the number of edges (i, j) for which the sign changes: $v(i)v(j) < 0$. For a general $n \times n$ real symmetric matrix H, its supporting graph G is defined as the graph with n vertices and (ij) is an edge whenever $H_{ij} \neq 0$ and $i \neq j$. If we order the eigenvalues of H, $\lambda_1(H) \leq \ldots \leq \lambda_n(H)$, the *nodal edge count* of the k-th eigenvector v is the number of pairs $i < j$ such that $v(i)v(j)H_{ij} > 0$, which coincides with the special case of the discrete Laplacian.

As with quantum graphs, the nodal edge count is $k-1$ plus a *nodal surplus* bounded between 0 and β, the first Betti number of the graph. In the case of quantum graphs $k \to \infty$ while β is fixed, while here $k \leq n$ and β is often much larger, comparable[1] with n^2. One might guess that the nodal surplus concentrates around $\beta/2$, symmetrically, so that its sum over all eigenvectors is $n\beta/2$. However, in a forthcoming work, we provide extreme examples for every n and $\beta \leq \binom{n-1}{2}$ where the sum of the nodal surplus is as small as β and as large as $(n-1)\beta$, which we prove to be the sharp general bounds for the nodal surplus sum.

On the other hand, our initial guess is justified when we consider the sum over all the signings of the graph (changing the signs of the off-diagonal entries of H symmetrically). Numerical studies suggests that this distribution tends to a Gaussian around $\beta/2$. In a separate work, we proved that this distribution is, indeed, precisely binomial with mean $\beta/2$, in the case of a complete graph and operators with sufficiently large potential.

The proofs of these results are based on a remarkable relation between the nodal surplus of an eigenvector and the Morse index of the corresponding eigenvalue with respect to magnetic perturbations. H is perturbed by a real anti-symmetric matrix α (often called magnetic potential) as follows $(H_\alpha)_{j\ell} = H_{j\ell}e^{i\alpha_{j\ell}}$. Gauge symmetry of H_α reduces the dependence of the eigenvalues of H_α to β dimensions, the first Betti number of the graph. The k-th eigenvalue $\lambda_k(H_\alpha)$ may be treated as a function on a β-dimensional torus, with critical points at H and its signings. If one assumes that λ_k is simple and $v(i) \neq 0$ for all i then each such critical point is non-degenerate with **Morse index equal to the nodal surplus** of the associated eigenvector v. We showed that if the eigenvector does vanish at a vertex, then the critical point is degenerate, and it lies on a non-singular critical manifold which has the topology of a well-studied space of planar linkages.

[1] β can be as large as $\binom{\{n-1}{2\}}$ which is the case for the complete graph.

1 Fourier Quasicrystals and Lee-Yang (stable) polynomials

Poisson's summation formula says that if $\Lambda \subset \mathbb{R}$ is a discrete periodic set, then the sum $\sum_{x\in\Lambda} f(x)$ for a "nice" function (Schwartz class) is proportional to the sum of the Fourier transform $\sum_{k\in\widehat{\Lambda}} \hat{f}(k)$ over the dual set $\widehat{\Lambda}$. The question of whether non-periodic summation formulas exists puzzled many mathematicians and relates to many fields. Early examples were given by Kahane and Mandelbrojt [29] and Guinand [27]. To frame the question, given a discrete set Λ with complex-coefficients (weights) $\{a_x\}_{x\in\Lambda}$, the atomic measure $\mu = \sum_{x\in\Lambda} a_x\delta_x$ is called a *Fourier Quasicrystals (FQ)*[2] if there exists another discrete set S complex-coefficients (weights) $\{c_k\}_{k\in S}$ such that

$$\sum_{x\in\Lambda} a_x f(x) = \sum_{k\in S} c_k \hat{f}(k) \quad \text{for all "nice" } f,$$

and also $\sum_{x\in\Lambda} |a_x| f(x)$ and $\sum_{k\in S} |c_k| \hat{f}(k)$ are finite. Although examples of non-periodic FQs were found, the question of finding a non-periodic FQ with unit coefficients $a_x \equiv 1$ and uniformly discrete support Λ remained open until recently. Kurasov and Sarnak provided such an example [32], based on the trace formula for quantum graphs. In this work, they provided a novel construction of FQs. A polynomial $p(z_1, \ldots, z_n)$ is called *Lee-Yang*, or stable on the inner-disc $\mathbb{D}$ and outer-disc $\mathbb{C} \setminus \overline{\mathbb{D}}$, if p has no zeros in $\mathbb{D}^n \cup (\mathbb{C} \setminus \overline{\mathbb{D}})^n$.

Theorem 1 (Kurasov, Sarnak)**.** *Given a Lee-Yang polynomial $p(z_1 \ldots, z_n)$ and positive frequencies $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n_+$, let Λ be the zero set of $F(x) = p(e^{ix\omega_1}, \ldots, e^{ix\omega_n})$ with multiplicities a_x. Then the zeros counting measure $\mu_{p,\omega} = \sum_{x\in\Lambda} a_x\delta_x$ is an FQ.*

This construction is visualized in Figure 1, showing the intersection of the line $t \mapsto (t\omega_1, t\omega_2)$ with the zero set $\{(x, y) : p(e^{ix}, e^{iy}) = 0\}$, for a two-dimensional choice of positive ω and Lee-Yang p. Based on a result of Olevskii and Ulanovskii [34], we proved in [7] an inverse result:

Theorem 2 (**A.**, Cohen, Vinzant)**.** *Every FQ with positive integer weights $a_x \in \mathbb{N}$ has the form $\mu_{p,\omega}$ for some $n \in \mathbb{N}$, Lee-Yang polynomial $p(z_1 \ldots, z_n)$ and positive frequencies $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n_+$ that are $\mathbb{Q}$-linearly independent.*

From here on, when writing $\mu = \mu_{p,\omega}$ we assume that p is Lee-Yang and ω has positive $\mathbb{Q}$-linearly independent entries. In [9], following [31], we consider the decomposition of p into distinct irreducible (Lee-Yang) polynomials $p = \prod_{j=1}^N q_j^{c_j}$, and show how it determines whether μ is periodic or not.

Theorem 3 (**A.**, Vinzant)**.** *If $\mu = \mu_{p,\omega}$ as above, then $\mu = \sum_{j=1}^N c_j \mu_{q_j,\omega}$. If q_j has only two monomials, then $\mu_{q_j,\omega}$ is periodic. Otherwise, $\mu_{q_j,\omega}$ is supported on a "very non-periodic" set Λ_j: it intersects any set of finite $\mathbb{Q}$-linear dimension in at most finitely many points (with uniform bounds)*

$$|\Lambda_j \cap A| \leq C(q_j, m), \ \textit{for every } A = \mathrm{span}_{\mathbb{Q}}\{x_1, x_2, \ldots, x_m\}.$$

Given $n \in \mathbb{N}$ and $d \in \mathbb{N}^n$ let $\mathrm{LY}(n, d)$ denote the space of Lee-Yang polynomials in n variables with degree d_j in z_j for all j.

Theorem 4 (**A.**, Vinzant)**.** *For any $n \geq 2$ and $d \in \mathbb{N}^n$ there is a generic*[3] *subset in $\mathrm{LY}(n, d)$ of polynomials p such that for any positive $\mathbb{Q}$-linearly independent ω the FQ $\mu_{p,\omega}$ is "very non-periodic" with unit coefficients and uniformly discrete support.*

[2]The name quasicrystal comes from crystallography, the experimental science of discerning the arrangement of atoms in crystals. The word crystal refer to a solid material with periodically ordered atomic structure. The atomic structure is measured indirectly through its Fourier transform by diffraction experiments. The diffraction of a crystal has sharp picks, while unordered atomic structures have a continuous "smeared" diffraction pattern. Surprisingly, materials with non-periodic atomic structure that exhibit a diffraction with sharp picks were found [37], and were called quasicrystals. These correspond to non-periodic (three-dimensional) summation formulae, but the support of the Fourier transform is usually dense, unlike Fourier Quasicrystals.

[3]An open, dense subset with full measure, whose complement has lower dimension.

We show in [9] that FQs with positive integer weights have well defined gap distributions. Writing $\mu_{p,\omega} = \sum_{j\in\mathbb{Z}} x_j \delta_{x_j}$ where $\{x_j\}_{j\in\mathbb{Z}} = \Lambda$ are the zeros of $F(x) = p(e^{ix\omega_1}, \ldots, e^{ix\omega_n})$ numbered increasingly with multiplicity, the atomic measure behaves like a random point processes with random i.i.d gaps $x_{j+1} - x_j \sim \rho$ for some gap distribution $\rho = \rho(p, \omega)$.

Theorem 5 (**A.**, Vinzant)**.** *Let $\mu_{p,\omega} = \sum_{j\in\mathbb{Z}} x_j \delta_{x_j}$ as above. Then there exists a probability measure $\rho = \rho(p,\omega)$, supported on a finite interval $[0, R]$, such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(x_{j+1} - x_j) = \int f d\rho, \text{ for every continuous } f : \mathbb{R} \to \mathbb{C}.$$

If $\mu_{p,\omega}$ is periodic then ρ is a finite sum of atoms. If p is irreducible with more than two monomials then ρ is absolutely continuous. Otherwise it is the sum of an absolutely continuous measure and finitely many atoms.

A particular case of interest studied in [9] is the limit of $\rho(p,\omega)$ as $\omega \to \mathbf{1} = (1, 1, \ldots, 1)$ while remaining $\mathbb{Q}$-linearly independent. The limit $\nu(p, \mathbf{1})$ can be explicitly expressed as the empirical measure of simpler random distributions. In particular, motivated by Quantum Chaos, we provide examples where the limiting gap distributions are well known distributions:

1. **Poisson:** If $p = (1 - z_1) \cdots (1 - z_n)$ then $\nu(p, \mathbf{1})$ is the empirical measure of the gaps between n random points on the unit circle. It converges to the Poisson distribution as $n \to \infty$.

2. **CUE:** If $u \sim \mathrm{Haar}(U_n)$ is a random $n \times n$ unitary matrix and $p = (1 - \mathrm{diag}(z_1, \ldots, z_n)u)$, then p is a random Lee-Yang polynomial, and the empirical measure $\mathbb{E}\nu(p, \mathbf{1})$ is the gap distribution of the eigenvalues of a random $u \sim \mathrm{Haar}(U_n)$. This measure has an $n \to \infty$ limit which is called the CUE gap distribution.

1.0.1 Future work on one-dimensional FQs

One of my goals is to prove Smilansky's conjecture [12, 30] which says that the distribution of gaps between eigenvalues of certain quantum graphs tend to the limiting gap distribution of random GOE matrices, in some limit of large graphs. This can be formalized as convergence of gap distributions $\rho = \rho(p, \omega)$ as above, for a special type of Lee-Yang polynomials, as the numebr of variables tends to infinity. I intend to apply tools developed in this work to this problem.
In an ongoing project with an MIT undergraduate we used the explicit limit $\nu(p, \mathbf{1})$ to construct an efficient algorithm to estimate the gap distributions of $\mu_{p,\omega}$ when the $\mathbb{Q}$-linear dimensionof ω is $1 < d < n$. We analyzed the polynomial p of the graph in [30], which has $n = 6$ and for which we know that $\rho(p, \omega)$ is close to GOE when $d = 6$. We found that the distribution deviates from GOE as d decrease with a significant difference between the intermediate dimensions $d = 2, 3$ to $d = 4, 5$. I intend to further investigate this phenomena in greater generality, in hope of finding a critical dimension.

1.1 Future work on High Dimensional Fourier Quasicrystals

The Kurasov-Sarnak construction of one-dimensional FQs considers the intersection of a codimension one variety $\{\mathbf{z} \in \mathbb{C}^n : p(\mathbf{z}) = 0\}$ with a one-dimensional curve $\{\exp(ix\ell) : x \in \mathbb{C}\}$, where $\exp : \mathbb{C}^n \to (\mathbb{C}^*)^n$ is the coordinate-wise exponent. In an ongoing work we generalized this construction to higher dimensions. To do so we define the notion of a Lee-Yang variety $X \subset (\mathbb{C}^*)^n$ of a co-dimension c, by means of allowed sign changes of consecutive entries of $(\log |z_1|, \ldots, \log |z_n|)$ for any $(z_1, \ldots, z_n) \in X$.

Theorem 6 (**A.**, Kummer, Kurasov, Vinzant)**.** *Let $X \in (\mathbb{C}^*)^n$ be a Lee-Yang variety of codimension d, let $L \in \mathbb{R}^{n\times d}$ be a matrix with positive $d \times d$ minors, and let $\Lambda = \{\boldsymbol{x} \in \mathbb{C}^d : \exp(iL\boldsymbol{x}) \in X\}$. Then $\Lambda \subset \mathbb{R}^d$, is discrete, and $\mu = \mu(X, L) = \sum_{x\in\Lambda} \delta_x$ is a d-dimensional FQ.*

Theorem 7 (**A.**, Kummer, Kurasov, Vinzant)**.** *In the above construction, if X is irreduicoble and is not a torus coset, and if the entries of L are algebraically independent over $\mathbb{Q}$, then Λ is "very non-periodic" as in Theorem 3.*

In particular, any set contained in a projection of a high dimensional lattice can intersect Λ in at most finitely many points. This includes the set whose Voronoi diagram is the Penrose tiling*, or any other "cut and project" set.*

The Voronoi diagram helps visualize such FQs. A two-dimensional example is given in Figure 1, given by a certain choice of $L \in \mathbb{R}^{2\times 3}$ with positive minors, and a one-dimensional Lee-Yang variety $X \subset \mathbb{C}^3$. In this case, the torus part $X \cap \mathbb{T}^3$ has two connected components, so the cells in the diagram are colored according to the component this atom came from.

2 Spectral geometry of quantum graphs (metric graphs)

A finite *Quantum graph*, or *metric graph*, (Γ, ℓ) is described by a graph $\Gamma = (V, E)$ of N edges $e_1, \ldots, e_N$, each identified with a line segment $e_j = [0, \ell_j]$, where $\ell = (\ell_1, \ldots, \ell_n)$ is the vector of edge lengths. In this way a function f on (Γ, ℓ) is described by its restrictions to edges $f|_{e_j} : [0, \ell_j] \to \mathbb{C}$, and the Laplacian Δ acts edgewise by $(\Delta f)|_{e_j} = -f|''_{e_j}$ on functions that satisfy suitable (Neumann-Kirchhoff) vertex conditions. By eigenvalues and eigenfunctions of (Γ, ℓ) we mean those of the Laplacian. We number the (square-root) eigenvalues increasingly including multiplicity,

$$0 = k_0 < k_1 \leq k_2 \leq k_3 \ldots \nearrow \infty.$$

As a remark, this model can be generalized by changing vertex cinditions, and adding magnetic and electric potentials, see [16] for a thorough review on quantum graphs. Quantum graphs appeared in various scientific disciplines in the last few decades, modeling complex phenomena such as superconductivity in granular and artificial materials [1], acoustic and electromagnetic wave-guide networks [13] and Anderson localization [19,36] to name but a few. However, the "simple" model described above already serves as a good one-dimensional model for spectral geometry and quantum chaos, as discussed in [25]. For this end, we consider large graphs and $\mathbb{Q}$-linearly independent edge lengths. The graph is associated with an $2N \times 2N$ real orthogonal matrix S_Γ, and a (Lee-Yang) polynomial $P_\Gamma(z_1, \ldots, z_n) = \det(1 - \operatorname{diag}(z_1, \ldots, z_n, z_1, \ldots, z_n)S)$ such that the spectrum of (Γ, ℓ) correspond to the zeros of $k \mapsto p(e^{ik\ell_1}, \ldots, e^{ik\ell_N}) = p(\exp(ik\ell))$. The eigenfunction f corresponding to the eigenvalue k^2 is described by $\mathbf{a}$, an eigenvector of the unitary matrix $\operatorname{diag}(\exp(ik\ell), \exp(ik\ell))S$ with eigenvalue 1.[4] This observation was made independently in [30, 35] and led to the trace formula for quantum graphs, to spectral statistics computations [12,30] and many other applications. Recently, Kurasov and Sarnak [31] showed that P_Γ is irreducible (except for certain specific graphs) which allowed to classify arithmetic properties of the spectrum [31, 32].

2.1 Universal nodal count for metric graphs

Let $\phi(n)$ denote the number of zeros of the n-th eigenfunction of a given quantum graphs (Γ, ℓ). It is called the *nodal count*[5] and it was proven in [14, 26] to be bounded between $n - 1 \leq \phi(n) \leq n - 1 + \beta$, assuming the n-th eigenvalue k_n is simple and its eigenfunction f_n does not vanish on vertices. In [2, 3] we show that these are generic conditions. It was conjectured in [26] that the bounded fluctuation $\sigma(n) = \phi(n) - (n - 1)$ should obey a universal behavior, for large enough and highly connected graphs with $\mathbb{Q}$-linearly independent lengths. The fact that the nodal surplus sequence $\{\sigma(n)\}_{n\in\mathbb{N}}$ obeys a well defined law was shown [2]. For simplicity in all theorems to follow, unless stated other wise, the graph Γ is assumed to have no vertices of degree two and no edge connecting a vertex to itself.

Theorem 8 (**A.**, Band, Berkolaiko)**.** *Given a graph Γ with first Betti number β and $\mathbb{Q}$-linearly independent ℓ, there is a random variable $\sigma = \sigma(\Gamma, \ell)$ taking values in $\{0, 1, \ldots, \beta\}$, symmetric around its mean which is $\beta/2$, such that*

$$\lim_{n\to\infty} \frac{|\{m \leq n : \sigma(m) = j\}|}{N} = P(\sigma = j), \quad \textit{for all} \;\; j \in \{0, \ldots, \beta\}.$$

[4]The restriction of f to e_j and the entries a_j and a_{j+N} of $\mathbf{a}$ are related by $f|_{e_j}(t) = a_j e^{-ikt} + a_{j+N} e^{-ik(\ell_j - t)}$ for $t \in [0, \ell_j]$.

[5]For an eigenfunction with high enough eigenvalue, the number of zeros and the number of nodal domain differ exactly by β, so the fluctuation statistics is the same.

The proof relies on two previous results. A characterization of σ_n as the Morse index of the n-th eigenvalue as a function of magnetic perturbation of the Laplacian [18], and an ergodicity argument given in [12]. For a certain family of graphs we were able to exploit certain symmetry of the magnetic characterization to calculate the distribution of σ. We say that Γ has *disjoint cycles* if any pair of distinct simple cycles in Γ are disjoint (not sharing any vertex).

Theorem 9 (**A.**, Band, Berkolaiko). *If Γ is a graph with disjoint cycles and ℓ is $\mathbb{Q}$-linearly independent ℓ, then $\sigma \sim \mathrm{Bin}(\beta, \frac{1}{2})$. In particular, $\mathbb{E}\sigma = \beta/2$, $\mathrm{Var}\sigma = \beta/4$, and*

$$\frac{\sigma - \mathbb{E}\sigma}{\sqrt{\mathrm{Var}\sigma}} \to Z, \quad Z \sim N(0,1), \quad \textit{for} \;\; \beta \to \infty.$$

In [5], we show that the distribution of $\sigma = \sigma(\Gamma, \ell)$ is convex in ℓ. Based on this fact and the ergodicity argument, we constructed an efficient algorithm that calculates the maximal deviation (kolmogorov-smirnov distance) of $\frac{\sigma - \mathbb{E}\sigma}{\sqrt{\mathrm{Var}\sigma}}$ from $N(0,1)$ among all possible $\mathbb{Q}$-linearly independent ℓ's. We sampled 25 different types of graphs, some chosen at random and some are deterministic. The decrease of the maixmal deviation as β grows was clearly shown, uniformly, regardless of any other graph property. This led us to the following strong conjecture.

Conjecture 1 (**A.**, Band, Berkolaiko). *The maximal deviation between $\sigma(\Gamma, \ell)$ and the Gaussian distribution of same mean and variance goes to zero as $\beta \to \infty$, uniformly over all (Γ, ℓ) with first Betti number β and $\mathbb{Q}$-independent ℓ. Moreover, the variance of $\sigma(\Gamma, \ell)$ is of order β.*

We also prove in [5] that the conjecture holds for two more families of graphs.

2.1.1 Future work and questions

Some questions I intend to work on in the future, in the context of the universal distribution of the nodal surplus, include the nodal surplus distribution on sub-graphs, finding models of random graphs for which $\sigma(\Gamma, \ell)$ can be estimated, applying the method of [5] to concatenation of graphs, and controlling the correlation between the contributions of different cycles.

2.2 Neumann count

The concept of a *Neumann partition* was introduced independently in [33, 38], in analogy to nodal partitions of manifolds, and was further developed in [10, 11]. While for manifold the Neumann partition of the manifold according to an eigenfunction is defined via its Morse data, the case of a quantum graphs is simpler. The name "Neumann domain/partition" is due to the fact that the restriction of an eigenfunction f to its Neumann domain Ω is itself a Laplacian eigenfunction on Ω with Neumann domain boundary conditions. In [4, 6] we define the n-th Neumann partition of the quantum graph by removing the critical points of the n-th eigenfunction. We denote the Neumann count by μ_n, namely the number of connected components of $\{x \in (\Gamma, \ell) : f_n'(x) \neq 0\}$, which for high enough eigenvalues is equal to the number of critical points up to a constant shift. We always assume that the n-th eigenvalue is simple and the eigenfunction f_n has no vanishing derivatives at any vertex. In [3] I proved that these are generic conditions. In [4], we provide topological upper and lower bounds on the Neumann count:

$$n + 2 - 2\beta - |\partial\Gamma| \leq \mu_n \leq n + \beta,$$

where $|\partial\Gamma|$ is the number of degree 1 vertices of Γ. We prove an analog of Theorem 8, showing that the fluctuations $\mu_n - n$ have well defined distribution which is symmetric around its mean $|\partial\Gamma|/2 - 1$. This has applications for *inverse problems*, since the mean of this distribution gives $|\partial\Gamma|$, which together with the first Betti number β (obtained from the mean of the nodal surplus) provides an upper bound on the number of vertices and edges.

Similarly to the nodal count, the fluctuations of the Neumann count seem to obey a universal Gaussian behavior as well, from numerical simulation, which is uncorrelated with the nodal surplus distribution. The limit in this case is when $\beta + |\partial\Gamma| \to \infty$. This includes trees, for which the nodal surplus is identically zero.

Theorem 10 (**A.**, Band)**.** *[4] If Γ is a $(3,1)$-regular finite tree and ℓ is $\mathbb{Q}$-independent, then $\nu(n)-n-2$ has a Binomial distribution with parameters $Bin(|\partial\Gamma|-2,\frac{1}{2})$. In particular, it converges to a Gaussian as $|\partial\Gamma|\to\infty$.*

2.2.1 Future work and questions

Some further questions regarding the Neumann count are whether there exist distinct graphs with the same nodal and Neumann count? Is there a Neumann count analog to the nodal-magnetic theorem [18]? Numerically, it seems that the bounds $-\beta\leq\omega_n\leq 2\beta+|\partial\Gamma|-2$ are not optimal. We conjecture in [4] that better bounds hold $0\leq\omega_n\leq\beta+|\partial\Gamma|-2$.

2.3 Generic properties of eigenvalues and eigenfunctions

Fixing a graph Γ of N edges and letting the edge-lengths ℓ change in $\mathbb{R}_+^N$, we can consider the eigenpairs $(k_n,f_n)_{n=1}^{\infty}$ as functions of ℓ. Previous works of Friedlander [24] and Berkolaiko and Liu [17] showed that there is a Baire-generic set of edge lengths $\mathcal{G}\subset\mathbb{R}_+^N$ such that all eigenvalues of (Γ,ℓ) with $\ell\in\mathcal{G}$ are simple with eigenfunctions that do not vanish on vertices. In [3] I show that by restriction to a possibly smaller generic set, we may take $\mathcal{G}$ such that the derivatives of each eigenfunction are non vanishing on all vertices of degree >1. I further show that the set $\mathcal{G}$ is also of full Lebesgue measure and its complement is a countable union of sub-analytic sets of smaller codimension, and that we may take it such that the derivatives of each eigenfunction are non vanishing on all vertices of degree >1. We say in such case that $\mathcal{G}$ is *strongly generic*. We had shown in [2] that for any $\mathbb{Q}$-independent ℓ, the sequence of eigenpairs $(k_n,f_n)_{n\in\mathbb{N}}$ of (Γ,ℓ) has a density one subsequence of simple eigenvalues with eigenfunctions that do not vanish on vertices. In [3] I consider a much larger class of spectral properties. Given an eigenpair (k,f), with $k>0$, let $\mathrm{trace}_k(f)$ be the vector of values of f at the vertices, and the outgoing derivatives of f normalized by $\frac{1}{k}$. We consider two vertex values and two outgoing derivatives for each edge so $\mathrm{trace}_k(f)\in\mathbb{C}^{4N}$. Consider the notation $\exp(k\ell):=(e^{ik\ell_1},e^{ik\ell_2},\dots,e^{ik\ell_N})\in\mathbb{C}^N$.

Theorem 11 (**A.**)**.** *Given a graph Γ of N edges and a polynomial $q\in\mathbb{C}[z_1,\dots,z_{5N}]$. Assume that $q(\exp(k\ell),\mathrm{trace}_k(f))$ is homogeneous in the $\mathrm{trace}_k(f)$ coordinates. If there exists $\ell\in\mathbb{R}_+^N$ such that (Γ,ℓ) has a simple eigenvalue $k>0$ with eigenfunction f such that*

$$q(\exp(k\ell),\mathrm{trace}_k(f))\neq 0.$$

Then, there is a strongly generic set of ℓ's such that every eigenpair of (Γ,ℓ) has simple eigenvalue and satisfies

$$q(\exp(k\ell),\mathrm{trace}_k(f))\neq 0.$$

Moreover, for any $\mathbb{Q}$-independent ℓ, there is a density one subsequence of eigenpairs $(k_{n_j},f_{n_j})_{j\in\mathbb{N}}$, such that k_{n_j} is simple and

$$q(\exp(k_{n_j}\ell),\mathrm{trace}_{k_{n_j}}(f_{n_j}))\neq 0.$$

The main ingredient in the proof is that the polynomial P_Γ is irreducible, which was conjectured by Colin de Verdière [21], and proved by Kurasov and Sarnak [31]. One application of this theorem regards common eigenvalues of different graphs. Gutkin and Smilansky [28] showed that distinct metric graphs (Γ,ℓ) and (Γ',ℓ') with $\mathbb{Q}$-independent ℓ and ℓ' do not have the same spectrum. Theorem 11 may be applied to provide a stronger statement for the case where $\ell=\ell'$.

Corollary 12 (**A.**)**.** *If Γ and Γ' are distinct (non-isomorphic) graphs of N edges, then there is a strongly generic set of ℓ's for which the only common eigenvalue of (Γ,ℓ) and $(\Gamma',\ell'=\ell)$ is zero. Moreover, for any $\mathbb{Q}$-independent ℓ, there is a density one subsequence of (Γ,ℓ) eigenvalues which are not eigenvalues of $(\Gamma',\ell'=\ell)$.*

Future work:

In this line of research, my goal is to prove a a conjecture of Sarnak: For generic ℓ, the eigenvalues of (Γ, ℓ) are $\mathbb{Q}$-linearly independent.

Another goal in this context is "Quantum Unique Ergodicity" for large graphs with $\mathbb{Q}$-independent ℓ. This question, in the language introduced in [3], is whether all the trace vectors $\mathrm{trace}_{k_{n_j}}(f_{n_j})$ in the trace space have components of the order $1/N$, as $N \to \infty$. I believe some progress can be made by applying algebraic tools to my construction of the trace space.

3 Nodal count distribution of signed matrices

Let G be a simple connected graph with n vertices, E edges and denote its first Betti number by $\beta = E - n + 1$. Let H be a real symmetric $n \times n$ matrix supported on G with eigenvalues numbered increasingly $\lambda_1(H) \leq \cdots \leq \lambda_n(H)$. Assuming that $\lambda_k(H)$ is simple and its eigenvector v has $v(i) \neq 0$ for all i, *the nodal edge count* $\phi(H,k)$ is the number of pairs $i < j$ such that $v(i)v(j)H_{ij} > 0$. The *nodal surplus* $\phi(H,k) - (k-1)$ is bounded between 0 and β. See [22] for a review of the many works leading to the upper bound, and [15] for the lower bound. As in the case of quantum graphs we consider the distribution of the nodal surplus, and its mean, which we expect to be $\beta/2$. Equivalently, we expect the sum of $\phi(H,k) - (k-1)$ over k to be $n\beta/2$. However, in a forthcoming work we show that the sum can deviate drastically from $\beta/2$ with a distribution far from Gaussian.

Theorem 13 (**A.**, Urschel)**.** *If H as simple eigenvalues with non-vanishing eigenvectors, then the sum of the nodal surplus is bounded between β and $(n-1)\beta$, and these bounds are sharp.*
Moreover, for every n and $\beta \leq \binom{n-1}{2}$, we construct examples which attain these bounds. In particular, for any choice of $\beta < n-2$, we construct H with nodal surplus taking only 0 *and* 1 *values.*

On the other hand, numerical simulations suggests that if we consider the nodal surplus of all eigenvectors for all different *signings* of H, then the distribution does tend to Gaussian around $\beta/2$. We say that H' is a *signing* of H if it is real symmetric with $H'_{ii} = H_{ii}$ and $H'_{ij} = \pm H_{ij}$ for all i and j, and we denote the set of signings of H by $\mathcal{S}(H)$. Let us say that H is *regular* if it has only simple eigenvalues with non-vanishing eigenvectors. In a separate work [8] we show

Theorem 14 (**A.**, Goresky)**.** *If H has constant diagonal entries and every $H' \in \mathcal{S}(H)$ is regular, then*

$$\frac{1}{|\mathcal{S}(H)|} \sum_{H' \in \mathcal{S}(H)} \left(\frac{1}{n} \sum_{k=1}^{n} \phi(H', k) - (k-1) \right) = \beta/2$$

If the off-diagonal entries of H are non-positive, it is called a *discrete Schrodinger operator* on G. It has the form $H = L + V$ where L is a weighted Laplacian and V is a diagonal potential. Perturbing H with a magnetic potential α, a real anti-symmetric matrix, yields a discrete magnetic operator H_α, with $(H_\alpha)_{rs} = H_{rs} e^{i\alpha_{rs}}$. Changing α to $\alpha + df$, where $(df)_{ij} = f(i) - f(j)$, is called *gauge transformation* and results in the conjugation of H_α by a unitary diagonal matrix $\mathrm{diag}(e^{if(1)}, \ldots, e^{if(n)})$. The k-th eigenvalue, considered as a function $H_\alpha \mapsto \lambda_k(H_\alpha)$, is gauge-invariant and reduce to a function on the quotient space $\mathbb{T}(H)$, the space of gauge equivalence classes $[H_\alpha]$, which is β-dimensional torus. Berkolaiko [15] and Colin de Verdiere [20] showed if H is a discrete Schrodinger operator, $\lambda_k(H)$ is simple, and its eigenvector v is non-vanishing, $v(i) \neq 0$ for all i, then the eigenvalues function $\lambda_k : \mathbb{T}(H) \to \mathbb{R}$ has a non-degenerate critical point at $[H]$ with Morse index equal to $\phi(H,k) - (k-1)$. In [8] we generalize this result and characterize all possible critical points of λ_k.

Theorem 15 (**A.**, Goresky)**.** *The function $\lambda_k : \mathbb{T}(H) \to \mathbb{R}$ has exactly three types of critical points:*

1. *$[H_\alpha]$ for which $\lambda_k(H_\alpha)$ is a* ***multiple eigenvalue****.*
2. *$[H_\alpha]$ for which $\lambda_k(H_\alpha)$ is simple with non-vanishing eigenvector.*
   In this case $[H_\alpha]$ is a ***non-degenrate*** *critical point with* ***Morse index equal to the nodal surplus****, and it is the equivalence class of H or one of its signings.*

3. $[H_\alpha]$ *for which* $\lambda_k(H_\alpha)$ *is simple with eigenvector* v *that has* $v(i)=0$ *for a vertex* i *of degree* d_i. *In this case, under some generic conditions,* $[H_\alpha]$ *lies in a Morse Bott smooth critical manifold of dimension* $d_i - 3$.

The critical manifold in the last part is equal to a moduli space of planar polygons with fixed side lengths (or planar linkages), whose Betti numbers were computed in [23]. As a result of Morse inequalities we get the following sufficient condition for the nodal surplus distribution being binomial.

Theorem 16 (**A.**, Goresky)**.** *If* H_α *is regular for all* α, *then the distribution of the nodal surplus* $\phi(H',k)-(k-1)$ *across all eigenvalues of all signings* $H' \in \mathcal{S}(H)$ *is* $\mathrm{Bin}(\beta, \frac{1}{2})$.

Corollary 17 (**A.**, Goresky)**.** *Let* H *be supported on the complete graph and let* V *be diagonal with distinct diagonal elements. Then for large enough* $\eta > 0$, *the distribution of the nodal surplus of* $H + \eta V$ *and its signings is binomial as above.*

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