RESEARCH STATEMENT

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My research lies in the field of *mathematical physics*. My main subject of research is *spectral geometry of metric graphs*, also known as *quantum graphs*. I have also works in progress on different subjects. One work is on the gaps distribution of Fourier Quasicrystals, and another work on nodal count distribution of signed graphs via Morse theory of Bloch varieties. My research includes questions and methods that relate to other mathematical disciplines such as

(1) Graph theory.
(2) Statistics and Probability.
(3) Fourier Analysis.
(4) Stable Polynomials (real algebraic geometry).
(5) Dynamics and Ergodic theory.
(6) Analytic Number Theory.

The research statement has three sections. The first section provides some highlights from my works on the nodal count distribution for metric graphs [ABB18, ABB22] in collaboration with Ram Band and Gregory Berkolaiko, the work on Neumann count for metric graphs [AB21] in collaboration with Ram Band, and my recent work on generic Laplacian eigenvalues and eigenfunctions on metric graphs [Alo22]. The second section provides some results from the work in progress on gaps distribution of Fourier Quasicrystals [AV] in collaboration with Cynthia Vinzant. The third section provides some results from the work in progress on nodal count distribution for signed graphs via Morse theory of Bloch varieties [AG] in collaboration with Mark Goresky. Each of these sections contains a relevant future work subsection. Before discussing metric graphs, let me introduce two open problems in spectral geometry and quantum chaos, that serve as my motivation.

0.1. **Quantum Chaos Motivation.** Consider the "simple" case of Laplacian with Dirichlet boundary conditions on a planar domain $\Omega \subset \mathbb{R}^2$. Let $N_\Omega(\lambda)$ be the number of eigenvalues smaller than λ. Hermann Weyl [Wey11] calculated the asymptotic growth $N(\lambda) \sim (\mathrm{area}(\Omega)/4\pi)\,\lambda$. A finer analysis of the fluctuations around the asymptotic growth is much harder. Consider the gaps between consecutive eigenvalues for example. A famous open conjecture says (roughly)

Conjecture 0.1. [BGS84] *If the "billiard flow" on a planar domain is **chaotic**, then the (properly normalized) gaps distribution is **universal**, and agrees with that of GOE random matrices.*

This conjecture was motivated by works of Wigner and was first suggested by Berry, see [Ber87] and the references therein. Counter examples for more general chaotic manifolds were found by Sarnak [Sar93]. Another conjecture by Berry was that the eigenfunctions of a chaotic system should have, locally, a universal behavior of a random Gaussian field (now called the Berry field). Here too, counter examples were found by Sarnak and Rudnick [RS94]. Another example of an eigenfunction's property that is believed to fluctuate in a universal manner is the *nodal count*. The nodal count ν_n of the n-th eigenfunction f_n is the number of connected components of $\{x \in \Omega : f_n(x) \neq 0\}$. Blum-Gnutzmann-Smilansky [BGS02] and Bogomolni-Schmit [BS02] conjectured:

Conjecture 0.2. [BGS02, BS02] *Given a* ***chaotic*** *domain in the plane, its nodal count* ν_n *is distributed as a Gaussian with mean and variance of order* n.

This conjecture is still open. The conjecture was motivated by analysis of the nodal count of random waves as a percolation model [BS02]. A rigorous analysis of the mean and variance of the nodal count fluctuations for random waves (Berry's Gaussian field) was made by Nazarov and Sodin [NS09, NS20]. The percolation of the nodal domains of such a Gaussian field was recently addressed by Duminil-Copin, Rivera and others [DCRRV21, MRVKS20].

1. Spectral geometry of metric graphs (quantum graphs)

A *metric graph*, which we denote by (Γ, ℓ), is a 1D manifold with singularities. The graph structure Γ represents the singular points as vertices and the smooth parts as edges between vertices. We will assume that Γ is a finite connected simple graph with N edges and without vertices of degree two (removable singularities). Every edge e_j has length ℓ_j, given by the edge lengths vector $\ell = (\ell_1, \ldots, \ell_N)$. Given a function f on (Γ, ℓ), its restriction to e_j is a function on $[0, \ell_j]$. The Laplacian acts edgewise as (minus) second derivative and we restrict to functions that satisfy Neumann-Kirchhoff vertex conditions. By eigenvalues and eigenfunctions of (Γ, ℓ) we mean those of the Laplacian. We number the (square-root) eigenvalues increasingly including multiplicity,

$$0 = k_0 < k_1 \leq k_2 \leq k_3 \ldots \nearrow \infty.$$

This model, and its generalization to any Schrödinger operator acting along the edges, appeared in various scientific disciplines in the last few decades, modeling complex phenomena such as superconductivity in granular and artificial materials [Ale83], acoustic and electromagnetic wave-guide networks [BK03] and Anderson localization [CMV06, SS00] to name but a few. We will only consider the "simple" geometric setting of Laplacian (without magnetic or electric potential) and Neumann-Kirchhoff vertex conditions. This model already serves as non-trivial one-dimensional model for spectral geometry, where exotic mathematical phenomena can often occur. For example, some graph structures have frequent appearance of scars (unusual localization of eigenfunctions) [BKW04, CdV15]. Another example, number theoretic in nature, is a recent work in progress by Kurasov and Sarnak on the arithmetic structure of the spectrum of a metric graph. They show that if the entries of ℓ are linearly independent over $\mathbb{Q}$ (we say $\mathbb{Q}$-independent), then the spectrum is infinite dimensional over $\mathbb{Q}$ and has a bound on the maximal length of arithmetic progressions in it. The name "*quantum graph*" first appeared in the work of Kottos and Smilansky [KS97] where they suggested that a metric graph (Γ, ℓ) with $\mathbb{Q}$-independent ℓ is a good paradigm for quantum chaos. They based their argument on numerical experiments showing that the gaps distribution of the eigenvalues agrees with that of the GOE ensemble, as predicted by Conjecture 0.1 for chaotic systems. Barra and Gaspard [BG00] gave a description of the spectrum of a metric graph using an ergodic map on a hypersurface in a high-dimensional torus. Using which they expressed the gap distribution in an analytic implicit way, that enabled a better numerical calculation and showed a significant deviation from the GOE distribution. However, they conjectured that this deviation should go to zero as the graph structure grows. more appropriate paradigm for chaos was suggested - **growing sequence of metric graphs with $\mathbb{Q}$-independent lengths**.

1.1. **Universal nodal count for metric graphs.** Conjecture 0.2 says that for a chaotic domain $\Omega \subset \mathbb{R}^2$, there exists constants C, c (that depend on Ω) such that the nodal count ν_n can be written as

$$\nu_n = Cn + c\sqrt{n}\sigma_n,$$

and the normalized deviation σ_n has the statistics of a normal random variable,

$$\lim_{N\to\infty} \frac{|\{n \le N \,:\, \sigma_n \le t\}|}{N} = P(Z \le t), \qquad Z \sim N(0,1).$$

However, it is not even known whether the limits on the left-hand side exist. A similar universal behavior seems to occur for large metric graphs as well, and in that case much can be proven, as we showed in [ABB18, ABB22]. The nodal count ν_n for a metric graph (Γ, ℓ) is the number of connected components of $\{x \in (\Gamma, \ell) \,:\, f_n(x) \neq 0\}$. It is bounded by $n - \beta \le \nu_n \le n$ where β is the first Betti number of the graph. The upper bound was proved in [GSW04] and the lower bound in [Ber08]. Motivated by Conjecture 0.2, we consider the normalized deviation σ_n such that

$$\nu_n = n - \sigma_n.$$

We showed in [ABB18] that if ℓ is $\mathbb{Q}$-independent, then the σ_n deviations have well defined statistics,

$$\lim_{N\to\infty} \frac{|\{n \le N \,:\, \sigma_n = j\}|}{N} = P(\sigma = j),$$

for some distribution σ supported inside $\{0, 1, \dots, \beta\}$ that depends on (Γ, ℓ). The proof relies on two previous results. A characterization of σ_n as the Morse index of the n-th eigenvalue as a function of magnetic perturbation of the Laplacian [BW14], and an ergodicity argument given in [BG00] that uses the $\mathbb{Q}$-independence of ℓ to obtain analytic expressions for spectral averages. We show in [ABB18] that the distribution σ has mean $\beta/2$ and is symmetric around it. We call σ the *nodal surplus distribution* and denote it by $\sigma(\Gamma, \ell)$ to emphasize its (Γ, ℓ) dependence. We obtained $\sigma(\Gamma, \ell)$ for graphs whose simple cycles do not intersect at any vertex.

Theorem 1.1. [ABB18] *If Γ has vertex-disjoint cycles and ℓ is $\mathbb{Q}$-independent, then $\sigma(\Gamma, \ell)$ is Binomial with mean $\beta/2$ and variance $\beta/4$. By the Central Limit Theorem,*

$$\frac{\sigma - \beta/2}{\sqrt{\beta/4}} \to Z, \qquad Z \sim N(0,1).$$

In a recent work [ABB22], we formulate Conjecture 0.2 for metric graphs.

Conjecture 1.2. [ABB22] *The distance between $\sigma(\Gamma, \ell)$ and the Gaussian distribution of same mean and variance goes to zero as $\beta \to \infty$, uniformly over all (Γ, ℓ) with first Betti number β and $\mathbb{Q}$-independent ℓ.*

We also conjecture that the variance is of order β. A detailed and quantified statement is given in [ABB22], together with numerical experiments that validate the conjecture. In [ABB22] we prove the conjecture for two more families of graphs, we prove that $\sigma(\Gamma, \ell)$ is convex in ℓ and we use it to provide an implicit (sharp) upper bound, $C(\Gamma)$, on the difference between $\sigma(\Gamma, \ell)$ and the relevant Gaussian. This bound is uniform in ℓ, and can be numerically evaluated (efficiently). The experiments in [ABB22] calculate $C(\Gamma)$

numerically for 25 different types of graphs, including random and deterministic. The decrease of $C(\Gamma)$ as β grows is clearly shown.

1.1.1. *Future work and questions.*

(1) Relating the nodal surplus distribution of a graph to its sub-graphs.
(2) Finding models of random graphs for which $\sigma(\Gamma,\ell)$ can be estimated?
(3) Applying the method of [ABB22] to graphs with cluster of disjoint cycles, by replacing the CLT argument with other types of martingale CLT results.
(4) Controlling the correlation between the contributions of different cycles in terms of the number of common vertices.

1.2. **Neumann count.** The concept of a *Neumann partition* was introduced independently in [Zel13, MF14], in analogy to nodal partitions of manifolds, and was further developed in [BF16, BET17]. The Neumann partition is a Morse partition of the manifold according to an eigenfunction. Connected components of the Neumann partition are called Neumann domains. The name "Neumann domain/partition" is due to the fact that the restriction of an eigenfunction f to its Neumann domain Ω is itself a Laplacian eigenfunction on Ω with Neumann domain boundary conditions. We define a metric graph analog in [ABBE20, AB21]. The n-th Neumann partition of a metric graph (Γ,ℓ) is given by removing the critical points along the edges. We denote the Neumann count by μ_n, namely the number of connected components of $\{x\in(\Gamma,\ell)\,:\,f_n'(x)\neq 0\}$. In [AB21], we provide topological upper and lower bounds on the Neumann count:

$$n+2-2\beta-|\partial\Gamma|\leq\mu_n\leq n+\beta,$$

where $|\partial\Gamma|$ is the number of degree 1 vertices of Γ. Similarly to the nodal count, the deviation ω_n is defined by $\mu_n=n-\omega_n$. We show in [AB21] that then these deviations have well defined statistics when ℓ is $\mathbb{Q}$-independent,

$$\lim_{N\to\infty}\frac{|\{n\leq N\,:\,\omega_n=j\}|}{N}=P\left(\omega=j\right),$$

for some distribution ω supported on $\{-\beta,\ldots,2\beta+|\partial\Gamma|-2\}$. We also show that ω has mean $\frac{\beta+|\partial\Gamma|-2}{2}$ and is symmetric around its mean. Two applications for **inverse problems**:

(1) The nodal count and Neumann count provide different information on the graph structure. For example, $\sigma\equiv 0$ for every tree graph (since $\beta=0$), however ω can distinguish between trees of different $|\partial\Gamma|$, since $\mathbb{E}(\omega)=\frac{|\partial\Gamma|-2}{2}$.
(2) Access to both $\mathbb{E}(\omega)$ and $\mathbb{E}(\sigma)$ gives β and $|\partial\Gamma|$ which bound the possible number of edges and vertices of the graph structure.

Similarly to σ, we conjecture that the distribution of ω has a universal Gaussian behavior as $\beta+|\partial\Gamma|$ grows to infinity. Following [ABB18] we prove a binomial result. Call a graph $(d,1)$-regular if its vertex degrees are either d or 1.

Theorem 1.3. [AB21] *If Γ is a $(3,1)$-regular finite tree and ℓ is $\mathbb{Q}$-independent, then $\omega(\Gamma,\ell)$ is Binomial with parameters $Bin(|\partial\Gamma|-2,\frac{1}{2})$. In particular, it converges to a Gaussian as $|\partial\Gamma|\to\infty$.*

1.2.1. *Future work and questions.*

(1) Are there distinct graphs with the same nodal and Neumann count?
(2) Is there a Neumann count analog to the nodal-magnetic theorem [BW14].
(3) Numerically, it seems that the bounds $-\beta\leq\omega_n\leq 2\beta+|\partial\Gamma|-2$ are not optimal. We conjecture in [AB21] that better bounds hold $0\leq\omega_n\leq\beta+|\partial\Gamma|-2$.

1.3. **Generic properties of eigenvalues and eigenfunctions.** Fixing a graph Γ of N edges and letting the edge-lengths ℓ change in $\mathbb{R}_+^N$, we can consider the eigenpairs $(k_n, f_n)_{n=1}^\infty$ as functions of ℓ. Previous works of Friedlander [Fri05] and Berkolaiko and Liu [BL17] showed that there is a Baire-generic set of edge lengths $\mathcal{G} \subset \mathbb{R}_+^N$ such that all eigenvalues of (Γ, ℓ) with $\ell \in \mathcal{G}$ are simple with eigenfunctions that do not vanish on vertices. In [Alo22] I show that by restriction to a possibly smaller generic set, we may take $\mathcal{G}$ such that the derivatives of each eigenfunction are non vanishing on all vertices of degree > 1. I further show that the set $\mathcal{G}$ is also of full Lebesgue measure and its complement is a countable union of sub-analytic sets of smaller codimension, and that we may take it such that the derivatives of each eigenfunction are non vanishing on all vertices of degree > 1. We say in such case that $\mathcal{G}$ is *strongly generic*. We had shown in [ABB18] that for any $\mathbb{Q}$-independent ℓ, the sequence of eigenpairs $(k_n, f_n)_{n\in\mathbb{N}}$ of (Γ, ℓ) has a density one subsequence of simple eigenvalues with eigenfunctions that do not vanish on vertices. In [Alo22] I consider a much larger class of spectral properties. Given an eigenpair (k, f), with $k > 0$, let $\mathrm{trace}_k(f)$ be the vector of values of f at the vertices, and the outgoing derivatives of f normalized by $\frac{1}{k}$. We consider two vertex values and two outgoing derivatives for each edge so $\mathrm{trace}_k(f) \in \mathbb{C}^{4N}$. Consider the notation $\exp(k\ell) := (e^{ik\ell_1}, e^{ik\ell_2}, \ldots, e^{ik\ell_N}) \in \mathbb{C}^N$.

Theorem 1.4. [Alo22] *Given a graph Γ of N edges and a polynomial $q \in \mathbb{C}[z_1, \ldots, z_{5N}]$. Assume that $q(\exp(k\ell), \mathrm{trace}_k(f))$ is homogeneous in the $\mathrm{trace}_k(f)$ coordinates. If there exists $\ell \in \mathbb{R}_+^N$ such that (Γ, ℓ) has a simple eigenvalue $k > 0$ with eigenfunction f such that*

$$q(\exp(k\ell), \mathrm{trace}_k(f)) \neq 0.$$

Then, there is a strongly generic set of ℓ's such that every eigenpair of (Γ, ℓ) has simple eigenvalue and satisfies

$$q(\exp(k\ell), \mathrm{trace}_k(f)) \neq 0.$$

Moreover, for any $\mathbb{Q}$-independent ℓ, there is a density one subsequence of eigenpairs $(k_{n_j}, f_{n_j})_{j\in\mathbb{N}}$, such that k_{n_j} is simple and

$$q(\exp(k_{n_j}\ell), \mathrm{trace}_{k_{n_j}}(f_{n_j})) \neq 0.$$

This theorem is based on a proof of Kurasov and Sarnak [KS] to the irreducibility conjecture of Colin de Verdière [CdV15]. One application of this theorem regards common eigenvalues of different graphs. Gutkin and Smilansky [GS01] showed that distinct metric graphs (Γ, ℓ) and (Γ', ℓ') with $\mathbb{Q}$-independent ℓ and ℓ' do not have the same spectrum. Theorem 1.4 may be applied to provide a stronger statement for the case where $\ell = \ell'$.

Corollary 1.5. [Alo22] *If Γ and Γ' are distinct (non-isomorphic) graphs of N edges, then there is a strongly generic set of ℓ's for which the only common eigenvalue of (Γ, ℓ) and $(\Gamma', \ell' = \ell)$ is zero. Moreover, for any $\mathbb{Q}$-independent ℓ, there is a density one subsequence of (Γ, ℓ) eigenvalues which are not eigenvalues of $(\Gamma', \ell' = \ell)$.*

Future work:

(1) A conjecture of Sarnak: For generic ℓ, the eigenvalues of (Γ, ℓ) should be $\mathbb{Q}$-independent.
(2) Quantum Unique Ergodicity for large graphs with $\mathbb{Q}$-independent ℓ. This question may be approached by analyzing the "trace space" introduced in [Alo22], to show that $\mathrm{trace}_{k_{n_j}}(f_{n_j})$ should have components of the order of $1/4N$, as $N \to \infty$.

2. Fourier Quasi-crystals and Stable Polynomials

A crystal, from a physical point of view, is a system of atoms in a lattice structure, such as metals and semi-conductors for example. The lattice structure can be experimentally observed by a scattering experiment, as the diffraction pattern would have peaks at the dual lattice locations. Mathematically, this phenomena is described by the *Poisson summation formula*. If Λ is a lattice with fundamental cell of volume one and dual lattice is S, then for any Schwartz function f with Fourier transform $\hat{f}$,

$$\sum_{x\in\Lambda} f(x) = \sum_{k\in S} \hat{f}(k).$$

Crystalline measures generalize the *Poisson summation formula*. A *Crystalline measure* is a discrete tempered distributions whose Fourier transform is also discrete. That is, a measure $\mu = \sum_{x\in\Lambda} a_x\delta_x$, supported on a discrete set $\Lambda \subset \mathbb{R}$, is a Crystalline measure if there is a discrete set $S \subset \mathbb{R}$ and complex coefficients $(c_k)_{k\in S}$ such that for any Schwartz function f, the following two infinite sums converge and are equal,

$$\sum_{x\in\Lambda} a_x f(x) = \sum_{k\in S} c_k \hat{f}(k). \tag{2.1}$$

Guinand provided some examples of non-periodic (and hence not Dirac Combs) Crystalline measures in [Gui59]. The convergence of the infinite sums in (2.1) may follow from cancellations of coefficients rather than the structure of Λ and S, as is the case for some of the Crystalline measures that Guinand constructed assuming the Riemann hypothesis. To distinguish between the types of convergence, Lev and Olevskii [LO15] call a crystalline measure μ a *Fourier Quasicrystal* (FQ) if the sums in (2.1) converge when replacing a_x and c_k with $|a_x|$ and $|c_k|$. A trivial FQ would be a Dirac Comb or a linear combination of such. The structure of non-trivial FQ is very rigid and suggests that these should be somewhat rare. For example, Lev and Olevskii [LO15] showed that given an FQ μ with a uniformly discrete support Λ and a uniformly discrete S, then μ is trivial. Recently, Kurasov and Sarnak [KS20] proved several open problems in this field by constructing a family of non-trivial FQ's with positive a_x coefficients and uniformly discrete support Λ (and S which is not uniformly discrete). They also provided an example of a such non-trivial FQ with positive coefficients whose support Λ is uniformly discrete. Let us call μ an $\mathbb{N}$-FQ if it is an FQ with uniformly bounded positive integer coefficients, $a_x \in \mathbb{N}$ and $a_x < M$ for some M. The construction of Kurasov and Sarnak provides $\mathbb{N}$-FQ's of the form

$$\mu = \sum_{x\in\mathbb{R}\ :\ F(x)=0} \deg(x)\delta_x,$$

where $\deg(x)$ is the zero's degree and F is a trigonometric polynomial $F(t) = p(\exp(it\ell))$, using the notation $\exp(it\ell) := (e^{it\ell_1}, \ldots, e^{it\ell_n})$. They showed in [KS20] that if the frequencies are positive $\ell \in \mathbb{R}_+^n$ and p is a *stable polynomial*, then μ is an $\mathbb{N}$-FQ. We call $p \in \mathbb{C}[z_1, z_2 \ldots z_n]$ a *stable polynomial*[1] if $p(z_1, z_2 \ldots z_n) \neq 0$ whenever all $|z_j| < 1$ or all $|z_j| > 1$. One can observe that if p is stable and $\ell \in \mathbb{R}_+^n$, then the zeros of $F(t) = p(\exp(it\ell))$ must be real. We say that F is a real-rooted trigonometric polynomial. Olevskii and Ulanovskii proved the following.

[1]A polynomial p is said to be "stable on Ω", for a domain $\Omega \subset \mathbb{C}$, if $p(z_1, z_2 \ldots z_n) \neq 0$ whenever $z_j \in \Omega$ for all j. For simplicity, we call p "stable" if it is stable on the disc $\mathbb{D}$ and its complement $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Theorem 2.1. [OU20] *A measure $\mu = \sum_{x\in\Lambda} a_x\delta_x$ is $\mathbb{N}$-FQ if and only if there exists a real-rooted trigonometric polynomial F such that Λ is its zero set and $a_x = \deg(x)$.*

In a work in progress [AV], joint with Cynthia Vinzant, we conjecture that all $\mathbb{N}$-FQ's can be constructed using stable polynomials as in [KS20].

Conjecture 2.2. [AV] *Any real rooted trigonometric polynomial $F(t)$ can be written as*

$$F(t) = ce^{it\omega}p(\exp(it\ell)),$$

*with constants $c \in \mathbb{C}, \omega \in \mathbb{R}$, a **stable** polynomial $p \in \mathbb{C}[z_1, z_2 \dots z_n]$ and a **positive** $\mathbb{Q}$-independent vector $\ell \in \mathbb{R}^n_+$, for some $n \in \mathbb{N}$.*

We call an $\mathbb{N}$-FQ μ a *Kurasov-Sarnak-FQ* if it can be constructed using stable polynomials as in [KS20] (so that the above conjecture states that any $\mathbb{N}$-FQ is Kurasov-Sarnak-FQ). We generalize a result of [KS20] and show that

Theorem 2.3. [AV, KS20] *Every Kurasov-Sarnak-FQ μ has an additive decomposition,*

$$\mu = \sum_{j=1}^{N} m_j\mu_j, \quad \text{with} \quad m_j \in \mathbb{N},\ \mu_j = \sum_{x\in\Lambda_j} a_{j,x}\delta_x,$$

for some N, such that every μ_j is a Kurasov-Sarnak-FQ and

1. *Either Λ_j is an arithmetic progression and $a_{j,x} = 1$ for all $x \in \Lambda_j$.*
2. *Or, μ_j is non-trivial and satisfies*
   1. *(a) $G \cap \Lambda_j$ is finite for any $G \subset \mathbb{R}$ which is a projection of a lattice in $\mathbb{R}^n$ to $\mathbb{R}$.*
   2. *(b) Almost all $a_{j,x}$ equal one,*

$$\lim_{T\to\infty} \frac{|\{x \in \Lambda_j \cap [-T,T]\ :\ a_{j,x} = 1\}|}{|\Lambda_j \cap [-T,T]|} = 1.$$

Both [KS20, OU20] construct Kurasov-Sarnak-FQ's with uniformly discrete support Λ. In [AV] we show that this is in fact a generic property.

Theorem 2.4. [AV] *A generic Kurasov-Sarnak-FQ has uniformly discrete support. Let $\mathbb{C}_d[z_1, \dots, z_n]$ be the vector space of polynomials of total degree at most d, and let $\mathbb{S}_{n,d} \subset \mathbb{C}_d[z_1, \dots, z_n]$ be the sub manifold of stable polynomials. There is an open-dense, full measure, algebraic set $O \subset \mathbb{S}_{n,d}$ such that for any $p \in O$ and any $\ell \in \mathbb{R}^n_+$, the zero set of $p(\exp(it\ell))$ is uniformly discrete.*

Finally, we discuss the gaps distribution for Kurasov-Sarnak-FQ's.

Theorem 2.5. [AV] *Let $p \in \mathbb{C}[z_1, \dots, z_n]$ be stable and let $\ell \in \mathbb{R}^n_+$ be $\mathbb{Q}$-independent. Let $(x_j)_{j\in\mathbb{Z}}$ denote the zeros of $F(x) := p(\exp(ix\ell))$, with multiplicity $\deg(x)$. Then,*

1. *The gaps distribution $\rho_{p,\ell}$ is well defined: For any $m \in \mathbb{Z}$ and continuous f,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=m+1}^{m+N} f(x_{n+1} - x_n) = \int f d\rho_{p,\ell}.$$

2. *The gaps distribution $\rho_{p,\ell}$ is a Borel probability measure supported inside a finite interval $[0, r]$. It has at most finitely many atoms and no singular-continuous part. If p is irreducible, then either all gaps are equal (arithmetic progression) or $\rho_{p,\ell}$ is absolutely continuous.*

(3) $\rho_{p,\ell}$ *is weakly continuous in* ℓ. *Given any continuous* f *and any sequence* $(\ell_j)_{j\in\mathbb{N}}$, *such that* $\ell_j \to \ell$ *and each* $\ell_j \in \mathbb{R}_+^n$ *is* $\mathbb{Q}$*-independent,*

$$\lim_{j\to\infty} \int f d\rho_{p,\ell_j} = \int f d\rho_{p,\ell}.$$

2.1. **Future work.**

(1) Applying tropical geometry tools to Conjecture 2.2.
(2) By analyzing the gap distribution of $\mu_{p,\ell}$ as a function of p and ℓ, we may be able to achieve progress in Conjecture 0.1 for metric graphs.
(3) Constructing a model of random Fourier Quasi-Crystals in terms of random stable polynomials. For example, if $p(z_1, \ldots, z_n) = \det(1 - \mathrm{diag}(z_1, \ldots, z_n)U)$ for a random unitary $U \in U_n$, then the gap distribution should converge to that of the CUE ensemble.
(4) Higher dimensional analogs. Is it possible to create quasi-crystals in higher dimensions using stable varieties of higher co-dimension?

3. Nodal count distribution of signed matrices

The work [AG] is a work in progress joint with Mark Goresky. It is partially an analog of [ABB22] for discrete graphs and partially extension of the works [Ber13, CdV13]. Let G be a simple connected graph on n ordered vertices labeled $1, 2, \cdots, n$. Write $r \sim s$ if $r \neq s$ are vertices connected by an edge. Functions on G are vectors, $v = (v_1, v_2, \cdots, v_n)$, in $\mathbb{R}^n$ or $\mathbb{C}^n$. An $n \times n$ matrix h is *supported on* G if $h_{rs} \neq 0 \implies r \sim s$ or $r = s$. We let $\mathcal{S}_n(G)$, $\mathcal{H}_n(G)$ and $\mathcal{A}_n(G)$ denote the vector spaces of real symmetric matrices, complex hermitian matrices and real antisymmetric matrices supported on G. A *discrete Schrödinger operator* is a real symmetric matrix $h \in \mathcal{S}_n(G)$ with $h_{rs} < 0$ for $r \sim s$. The quadratic form associated with $h \in \mathcal{S}_n(G)$ may be expessed as that of a weighted Laplacian Δ plus a "potential" V,

$$\langle f, hf \rangle = -\sum_{r\sim s} h_{rs} \left(f(r) - f(s)\right)^2 + \sum_{r=1}^{n} V(r) f(r)^2,$$

where $V(r) = h_{rr} + \sum_{r\sim s} h_{rs}$. We number the eigenvalues from below

$$\lambda_1(h) \leq \lambda_2(h) \leq \cdots \leq \lambda_n(h),$$

and consider each λ_k as a function of $h \in \mathcal{H}_n(G)$. Suppose λ_k is a simple (multiplicity one) eigenvalue of h with a nowhere-vanishing eigenvector v (meaning that $v_r \neq 0$ for all r). A basic problem in graph theory is to understand the behavior of the *nodal count* $\phi(h, k)$, that is, the number of edges $r \sim s$ for which v changes sign: $v(r)v(s) < 0$. It is known that $k - 1 \leq \phi(h, k) \leq k - 1 + \beta$, where β is the first Betti number of G. See [DGLS01] for a review of the many works leading to the upper bound, and [Ber13] for the lower bound. This motivates the definition of the *nodal surplus*

$$\phi(h, k) - (k - 1) \in \{0, 1, \cdots, \beta\}$$

and its probability distribution $P(h)$ over the n possible eigenvalues:

$$P(h)_s = \frac{1}{n} \#\{1 \leq k \leq n \,|\, \phi(h, k) - (k - 1) = s\}.$$

In numerical simulations for large graphs, this distribution seems to concentrate around $\frac{\beta}{2}$ with variance of the order of β, similarly to the observations for metric graphs in [ABB22].

Following [Ber13, CdV13, ABB18] we investigate the nodal count by considering *magnetic perturbations of h*. Given a discrete Schrödinger operator $h \in \mathcal{S}_n(G)$, a magnetic potential $\alpha \in \mathcal{A}_n(G)$ is a real anti-symmetric matrix supported on G and the associated *magnetic Schrödinger operator* $h_\alpha \in \mathcal{H}_n(G)$ is the Hermitian matrix $(h_\alpha)_{rs} = e^{i\alpha_{rs}} h_{rs}$. Given h we consider the manifold $\mathbb{T}_h \subset \mathcal{H}_n(G)$ of all h_α with $\alpha \in \mathcal{A}_n(G)$. $\mathbb{T}_h$ is a torus containing h. The k-th eigenvalue function λ_k, restricted to $\mathbb{T}_h$, has a critical point at h whose Morse index is exactly $\phi(h,k) - (k-1)$ [Ber13, CdV13]. The needed assumption is that $\lambda_k(h)$ is simple with a nowhere vanishing eigenvector. The critical point is non-degenerate if we consider λ_k as a function on the quotient $\mathcal{M}_h$, which is $\mathbb{T}_h$ modulo gauge transformations (that preserves the eigenvalues). The graphs of the eigenvalue functions $(h_\alpha, \lambda_k(h_\alpha))$, for $h_\alpha \in \mathbb{T}_h$ and $k = 1, 2, \ldots, n$, is also known as the *Bloch Variety* or *Dispersion relation manifold* associated to the periodic discrete Schrödinger operator $\hat{h}$ on $\widehat{G}$. For $\widehat{G}$, the universal abelian cover of G and $\hat{h}$ the periodic extension of h to $\widehat{G}$. The spectrum of $\hat{h}$ is the union of the spectrum of h_α for all $h_\alpha \in \mathbb{T}_h$, according to the Bloch theorem. It is a union of disjoint intervals called *bands* whose endpoints correspond to critical points on the Bloch Variety. The simplest critical points are the *symmetry points*: points fixed by complex conjugation, i.e. real symmetric matrices. These are all the *signed matrices*, i.e. the real symmetric matrices obtained from h by changing signs of off-diagonal elements. In [AG] we show that each critical point $g \in \mathbb{T}_h$ with nowhere vanishing eigenvector and simple eigenvalue $\lambda_k(g)$ is necessarily in the gauge equivalence class of a symmetry point, and we also give a homological criterion for symmetry points. We show that if g has a simple eigenvalue with eigenvector v, then g is a critical point if and only if $\bar{v}_r h_{rs} v_s \in \mathbb{R}$ for all $r \sim s$. Following [CdV13, Remark 1], we generalize the *nodal count* to any critical point g. We let $\phi(g,k)$ be the number of edges $r \sim s$ such that $\bar{v}_r h_{rs} v_s > 0$. We show that as long as the eigenvalue is simple and the eigenvector is nowhere vanishing then the *nodal surplus* $\phi(g,k) - (k-1)$ equals the Morse index, as before, and so the bounds remains between 0 and β, the first Betti number of G. The *nodal surplus distribution* $P(g)$ is similarly defined. Let $\mathcal{S}(h)$ stand for all the signed matrices obtained from $h \in \mathcal{S}_n(G)$. Assume that h and all its signings $h' \in \mathcal{S}(h)$ have simple eigenvalues with nowhere-vanishing eigenvectors, so the distribution can be averaged over signings to give the *average nodal distribution*

$$P(\mathcal{S}(h)) = 2^{-|E|} \sum_{h' \in \mathcal{S}(h)} P(h').$$

Following many numerical simulations, and based on joint works with Ram Band and Gregory Berkolaiko, we conjecture

Conjecture 3.1. [AG] *If G has large β and $h \in \mathcal{S}_n(G)$ satisfies the above assumptions so that $P(\mathcal{S}(h))$ is well defined, then $P(\mathcal{S}(h))$ is approximately binomial with mean $\beta/2$ and variance of order β.*

I will not elaborate at this point on the nature of this approximation or convergence to Gaussian. We have the following results.

Theorem 3.2. [AG] *Assume that h and all its signings $h' \in \mathcal{S}(h)$ have simple eigenvalues with nowhere-vanishing eigenvectors, and further assume that every critical point in $\mathbb{T}_h$, for every eigenvalue, has simple eigenvalue with nowhere-vanishing eigenvector. Then, $P(\mathcal{S}(h))$ is binomial with mean $\beta/2$ and variance $\beta/4$.*

Theorem 3.3. [AG] *Assume that h and all its signings $h' \in \mathcal{S}(h)$ have simple eigenvalues with nowhere-vanishing eigenvectors, and further assume that all diagonal elements of h are equal, then $P(\mathcal{S}(h))$ has mean $\beta/2$ and is symmetric around its mean.*

References

[AB21] Lior Alon and Ram Band. Neumann domains on quantum graphs. *Annales Henri Poincaré*, 22(10):3391–3454, Oct 2021. (document), 1.2, 1.3, 3

[ABB18] L. Alon, R. Band, and G. Berkolaiko. Nodal statistics on quantum graphs. *Communications in Mathematical Physics*, Mar 2018. (document), 1.1, 1.1, 1.2, 1.3, 3

[ABB22] Lior Alon, Ram Band, and Gregory Berkolaiko. Universality of nodal count distribution in large metric graphs. *Experimental Mathematics*, pages 1–35, 2022. (document), 1.1, 1.1, 1.2, 1.1, 3, 3

[ABBE20] Lior Alon, Ram Band, Michael Bersudsky, and Sebastian Egger. Neumann domains on graphs and manifolds. *Analysis and Geometry on Graphs and Manifolds*, 461:203–249, 2020. 1.2

[AG] Lior Alon and Mark Goresky. Nodal distribution of signed matrices. *In preperation.* (document), 3, 3.1, 3.2, 3.3

[Ale83] S. Alexander. Superconductivity of networks. A percolation approach to the effects of disorder. *Phys. Rev. B (3)*, 27(3):1541–1557, 1983. 1

[Alo22] Lior Alon. Generic laplace eigenfunctions on metric graphs. *Submitted to Journal d'Analyse Mathematique. arXiv:2203.16111*, 2022. (document), 1.3, 1.4, 1.5, 2

[AV] Lior Alon and Cynthia Vinzant. Fourier quasi-crystals and stable polynomials. *In preperation.* (document), 2, 2.2, 2.3, 2, 2.4, 2.5

[Ber87] Michael V Berry. Quantum chaology (the bakerian lecture). pages 307–322, 1987. 0.1

[Ber08] G. Berkolaiko. A lower bound for nodal count on discrete and metric graphs. *Comm. Math. Phys.*, 278(3):803–819, 2008. 1.1

[Ber13] G. Berkolaiko. Nodal count of graph eigenfunctions via magnetic perturbation. *Anal. PDE*, 6:1213–1233, 2013. preprint `arXiv:1110.5373`. 3

[BET17] Ram Band, Sebastian Egger, and Alexander Taylor. Ground state property of neumann domains on the torus. *arXiv preprint arXiv:1707.03488*, 2017. 1.2

[BF16] R. Band and D. Fajman. Topological properties of Neumann domains. *Ann. Henri Poincaré*, 17(9):2379–2407, 2016. 1.2

[BG00] F. Barra and P. Gaspard. On the level spacing distribution in quantum graphs. *J. Statist. Phys.*, 101(1–2):283–319, 2000. 1, 1.1

[BGS84] O. Bohigas, M. J. Giannoni, and C. Schmit. Characterization of chaotic quantum spectra and universality of level fluctuation laws. *Phys. Rev. Lett.*, 52(1):1–4, 1984. 0.1

[BGS02] G. Blum, S. Gnutzmann, and U. Smilansky. Nodal domains statistics: A criterion for quantum chaos. *Phys. Rev. Lett.*, 88(11):114101, 2002. 0.1, 0.2

[BK03] Noureddine Benchama and Peter Kuchment. An asymptotic model for wave propagation in thin high contrast 2D acoustic media. In *Progress in analysis, Vol. I, II (Berlin, 2001)*, pages 647–650. World Sci. Publ., River Edge, NJ, 2003. 1

[BKW04] G. Berkolaiko, J. P. Keating, and B. Winn. No quantum ergodicity for star graphs. *Comm. Math. Phys.*, 250(2):259–285, 2004. 1

[BL17] G. Berkolaiko and W. Liu. Simplicity of eigenvalues and non-vanishing of eigenfunctions of a quantum graph. *J. Math. Anal. Appl.*, 445(1):803–818, 2017. preprint `arXiv:1601.06225`. 1.3

[BS02] E. Bogomolny and C. Schmit. Percolation model for nodal domains of chaotic wave functions. *Phys. Rev. Lett.*, 88:114102, Mar 2002. 0.1, 0.2, 0.1

[BW14] G. Berkolaiko and T. Weyand. Stability of eigenvalues of quantum graphs with respect to magnetic perturbation and the nodal count of the eigenfunctions. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 372(2007):20120522, 17, 2014. 1.1, 2

[CdV13] Y. Colin de Verdière. Magnetic interpretation of the nodal defect on graphs. *Anal. PDE*, 6(5):1235–1242, 2013. 3

[CdV15] Y. Colin de Verdière. Semi-classical measures on quantum graphs and the Gauß map of the determinant manifold. *Annales Henri Poincaré*, 16(2):347–364, 2015. also `arXiv:1311.5449`. 1, 1.3

[CMV06] Kingwood Chen, Stanislav Molchanov, and Boris Vainberg. Localization on Avron-Exner-Last graphs. I. Local perturbations. In *Quantum graphs and their applications*, volume 415 of *Contemp. Math.*, pages 81–91. Amer. Math. Soc., Providence, RI, 2006. 1

[DCRRV21] Hugo Duminil-Copin, Alejandro Rivera, Pierre-François Rodriguez, and Hugo Vanneuville. Existence of an unbounded nodal hypersurface for smooth gaussian fields in dimension d $\geq$3. *arXiv preprint arXiv:2108.08008*, 2021. 0.1

[DGLS01] E. Brian Davies, Graham M. L. Gladwell, Josef Leydold, and Peter F. Stadler. Discrete nodal domain theorems. *Linear Algebra Appl.*, 336:51–60, 2001. 3

[Fri05] L. Friedlander. Genericity of simple eigenvalues for a metric graph. *Israel J. Math.*, 146:149–156, 2005. 1.3

[GS01] Boris Gutkin and Uzy Smilansky. Can one hear the shape of a graph? *J. Phys. A*, 34(31):6061–6068, 2001. 1.3

[GSW04] S. Gnutzmann, U. Smilansky, and J. Weber. Nodal counting on quantum graphs. *Waves Random Media*, 14(1):S61–S73, 2004. 1.1

[Gui59] AP Guinand. Concordance and the harmonic analysis of sequences. *Acta Mathematica*, 101(3-4):235–271, 1959. 2

[KS] P. Kurasov and P. Sarnak. Private communication. 1.3

[KS97] T. Kottos and U. Smilansky. Quantum chaos on graphs. *Phys. Rev. Lett.*, 79(24):4794–4797, 1997. 1

[KS20] Pavel Kurasov and Peter Sarnak. Stable polynomials and crystalline measures. *Journal of Mathematical Physics*, 61(8):083501, 2020. 2, 2, 2, 2.3, 2

[LO15] Nir Lev and Alexander Olevskii. Quasicrystals and poisson summation formula. *Inventiones mathematicae*, 200(2):585–606, 2015. 2

[MF14] Ross B. McDonald and Stephen A. Fulling. Neumann nodal domains. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 372(2007):20120505, 6, 2014. 1.2

[MRVKS20] Stephen Muirhead, Alejandro Rivera, Hugo Vanneuville, and Laurin Köhler-Schindler. The phase transition for planar gaussian percolation models without fkg. *arXiv preprint arXiv:2010.11770*, 2020. 0.1

[NS09] Fedor Nazarov and Mikhail Sodin. On the number of nodal domains of random spherical harmonics. *Amer. J. Math.*, 131(5):1337–1357, 2009. 0.1

[NS20] Fedor Nazarov and Mikhail Sodin. Fluctuations in the number of nodal domains. *Journal of Mathematical Physics*, 61(12):123302, 2020. 0.1

[OU20] Alexander Olevskii and Alexander Ulanovskii. Fourier quasicrystals with unit masses. *Comptes Rendus. Mathématique*, 358(11-12):1207–1211, 2020. 2.1, 2

[RS94] Zeév Rudnick and Peter Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Communications in Mathematical Physics*, 161(1):195–213, 1994. 0.1

[Sar93] Peter Sarnak. Arithmetic quantum chaos. *Blyth Lectures. Toronto*, 1993. 0.1

[SS00] Holger Schanz and Uzy Smilansky. Periodic-orbit theory of anderson localization on graphs. *Phys. Rev. Lett.*, 84:1427–1430, Feb 2000. 1

[Wey11] Hermann Weyl. Über die asymptotische verteilung der eigenwerte. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1911:110–117, 1911. 0.1

[Zel13] S. Zelditch. Eigenfunctions and nodal sets. *Surveys in Differential Geometry*, 18:237–308, 2013. 1.2

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